

On the Relationship between Thermodynamics and Special Relativity

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Abstract

Starting from a formulation for the dS element that includes movement, and considering the variation of the entropy Lorentz invariant, we found the relativistic transformations for thermodynamic systems that satisfy the three laws of thermodynamics. Particularly, we found the temperature and pressure transformations, given by $T' = \gamma T$ and $p' = \gamma^2 p$ respectively. Furthermore, we show that this transformations keeps the form of the state equation for an ideal gas in agreement with the relativity principle.

I. INTRODUCTION

Through history we have seen that many scientific discoveries have been reached starting from simple ideas. Nowadays the most part of research is based in the specialization of a single topic. This fact can be noted in the undergraduate courses in physics where is not usual to treat recent research themes related to the courses due to the complexity of those themes. One of the aims of this article is to show how theoretical research can be done by using simple ideas that can be learned in a undergraduated course.

In this article we treat a yet unsolved problem in theoretical physics: the contruction of a consistent relativistic thermodynamics theory. This particular problem have generated a long controversy in physics and can be approached by a student of a thermodynamics course, using only basic knowledge of thermodynamics and special relativity. Is in this way that including this particular topic as a part of an undergraduated thermodynamics or modern physics course could support the discussion of physics and motivate students to start scientific research at the undergraduate level.

Let us suppose a system A , which is in thermodynamic equilibrium, and two inertial frames, I and I' , where one of them (I) is at rest respectve to A and where the other (I') is moving with speed \mathbf{w} respectve to I . We orient the axes so the relative motion of the frames is in the x and x' directions. This is $\mathbf{w} = w\hat{x}$. The question is: how the thermodynamics quantities of A transform between I and I' ?

A century has passed since Einstein, in a first attempt to answer this question, stated that if the entropy is Lorentz invariant, and the two first laws of thermodynamics are fulfilled, then the temperature transforms as

$$T' = \frac{1}{\gamma}T, \quad (1)$$

where $\gamma = (1 - (w/c)^2)^{-1/2}$ and c is the speed of light¹; it means that a moving body appears colder. His temperature transformation was accepted for almost fifty years, as can be seen in^{2,3,4,5}.

In 1963, Ott⁶ affirmed that the temperature transform as

$$T' = \gamma T \quad (2)$$

by supposing the entropy as a Lorentz invariant, like other authors did (see for example⁹ and references therein). This result contradicts the previous affirmations because in this

treatment the body appears hotter.

A few years later, Landsberg^{7,8} stated that temperature should be Lorentz invariant but, years later, he restarted the problem by saying that it is impossible to obtain a general transformation of the temperature⁹.

Nowadays the question is still open. For example, in²¹, we can appreciate an answer derived from the formalism of statistical mechanics. Also, even when the topic is, in principle, theoretical, in this work the authors declare that certain astrophysical systems might give experimental data which allows us to decide what transformation is the correct one (see^{19,20} for more related works).

In this work we approach the problem using the formalism of thermodynamics. This must be equivalent to an approach from statistical mechanics since, due the postulate of Gibbs, thermodynamics quantities are averages of quantities obtained from statistical mechanics.

We want to emphasize that due the approach used to obtain the results in this work, it could be a good exercise to undergraduate students in physics, because basic knowledges in thermodynamics and special relativity are enough to understand the treatment used here. Also, the inclusion of this topic in a regular course of thermodynamics can generate scientific discussion in an actual theme.

II. TRANSFORMATION OF THERMODYNAMIC QUANTITIES

Our approach to the problem start by realizing that there is no reason to suppose that the expression

$$dS = \frac{1}{T}dU + \frac{p}{T}dV - \frac{\mu}{T}dN \quad (3)$$

has the same form for all inertial frames. Here T is the temperature, U the internal energy, p the pressure, V the volume, N the number of particles and μ the chemical potential. That entropy variation has not the same form in any inertial frame is because classical thermodynamics has not been dedicated to solve the problem about what occurs when there is a change between reference frames, since it has always considered the macroscopic properties of matter, but being at rest with respect to it.

Following a treatment similar to the one of Callen¹⁸, we write the dS element like an exact differential but also adding a new term that involves the relative movement between the thermodynamic system and the observer. The new term must depends of its fundamental

parameters: the total momentum \mathbf{P} of the thermodynamic system and the relative speed between the system and an inertial frame. In this case the velocity is \mathbf{w} .

The term that involves movement comes from the energy of a relativistic system. If the total mass of our thermodynamical system is M , then due the movement between the system and an inertial frame, we must add a kinetic energy E_m given by

$$E_m^2 = \mathbf{P}^2 + M^2. \quad (4)$$

Then,

$$dE_m = \frac{\mathbf{P}}{(\mathbf{P}^2 + m^2)^{1/2}} \cdot d\mathbf{P}, \quad (5)$$

and therefore

$$dE_m = \mathbf{w} \cdot d\mathbf{P}. \quad (6)$$

With this we can write the total variation of energy as

$$dE = dU + dE_m, \quad (7)$$

where E is the total energy of the system.

Later on, using (3) and (6), it follows

$$dE = TdS - pdV + \mu dN + \mathbf{w} \cdot d\mathbf{P}, \quad (8)$$

which gives a relativistic form of writing the element dS for a moving system:

$$dS = \frac{1}{T}dE + \frac{p}{T}dV - \frac{\mu}{T}dN - \frac{1}{T}\mathbf{w} \cdot d\mathbf{P}. \quad (9)$$

This form of the element dS includes the correction due to the inclusion of movement between frames. This generalization of Eq. (3) allow us to take into account the relativistic effects due to the relative motion when analyzing a thermodynamic system.

In this way, the entropy of a system must have the form $S = S(E, V, N, \mathbf{P})$, where the extensive quantities E , V , N and \mathbf{P} can be measured in any inertial frame. In this manner, the intensive variables are completely described, not only for the proper frame of the thermodynamic system, but also for any other inertial frame. In addition, as can be seen in Eq. (9), dS is an exact differential. Therefore we can define the temperature as

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{V, N, \mathbf{P}} \quad (10)$$

in any inertial frame, generalizing the well-known expression

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)_{V,N} . \quad (11)$$

In order to obtain the transformations of thermodynamic quantities between I and I' , we first assume that the variation of the entropy dS is Lorentz invariant, in agreement with all previous works cited here. This is,

$$dS = dS' \quad (12)$$

where S and S' are the entropies measured in the I and I' frames respectively. For the other quantities we have that the number of particles N is Lorentz invariant, and in this case, between I and I' , the volume V and the energy E transforms as

$$V' = \frac{1}{\gamma} V , \quad (13)$$

$$E' = \gamma E , \quad (14)$$

respectively, where E' and V' are measured in I' .

As the differential form for the entropy guarantees Eq. (10), using Eq. (12) and Eq. (14), it follows that the temperature T' , measured in the I' frame, is related to T by

$$\frac{1}{T'} = \left(\frac{\partial S'}{\partial E'} \right)_{V',N',\mathbf{P}'} = \frac{1}{\gamma} \left(\frac{\partial S}{\partial E} \right)_{V,N,\mathbf{P}} = \frac{1}{\gamma T} , \quad (15)$$

from where the temperature transformation

$$T' = \gamma T , \quad (16)$$

is obtained. This transformation is in agreement with the one by Ott⁶ and other authors^{10,11,12}. Also Eq. (16) is in agreement with the three laws of thermodynamics, unlike what was proposed by Einstein (see Eq (1)).

For the pressure p' , using Eq. (9), we have

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V} \right)_{E,\mathbf{P},N} , \quad (17)$$

Then, using Eq. (12), (13), (16) and (17) we obtain

$$p' = T' \left(\frac{\partial S'}{\partial V'} \right)_{E',\mathbf{P}',N'} = \gamma T \left(\frac{\partial S}{\partial (V/\gamma)} \right)_{E,\mathbf{P},N} . \quad (18)$$

This is, the pressure transforms as

$$p' = \gamma^2 p. \quad (19)$$

This result, previously obtained by Sutcliffe¹², is natural when we accept that Eq. (17) is the correct definition of pressure in any inertial frame.

Following the same argument, for the μ' potential,

$$\mu' = \gamma \mu, \quad (20)$$

it is obtained. If we note that μ is an energy, and energy transforms as γ , then, this μ transformation was expected.

As a example, from Eq. (13), Eq. (16) and Eq. (19), it follows that

$$\frac{pV}{Nk_B T} = \frac{p'V'}{N'k_B T'}, \quad (21)$$

which means that the equation of state of an ideal gas is Lorentz invariant. This means that, for any inertial observer, an ideal gas is still an ideal gas, regardless of relative inertial motion.

III. FINAL DISCUSSION

The development of relativistic thermodynamics has been a complex and hard to treat theme for a long time. There has been many proposals, many of them contradict each other. In this article we have formulated a theory of thermodynamics where the movement between inertial frames is considered. This led us to find a new expression for the entropy by considering the kinetic energy due to the relative motion between frames. The new entropy $S = S(E, V, \mathbf{P}, N)$ now depends on the total energy E instead of the internal energy U and also of the momentum of the system \mathbf{P} measured in a inertial frame. This choice allowed us to write the dS element as an exact differential that now has a new term, $(1/T)\mathbf{w} \cdot d\mathbf{P}$, which is related to the relative movement between frames. Using this new expression for dS , just as it is done in classical thermodynamics, we were able to define intensive thermodynamic quantities by taking partial derivatives of S respect to the extensive quantities. This formalism satisfies the three laws of thermodynamics and becomes the usual one in the limit $\mathbf{w} \rightarrow 0$.

Using this formalism we obtained the transformations given by Eq. (16), (19) and (20) for the temperature, the pressure and the chemical potential respectively. These transformations are completely general for any system that satisfy the three laws of thermodynamics.

Taking this last argument we were able to state why the temperature transformation given by Eq. (16) is the correct one in a relativistic thermodynamic theory. In thermodynamics the temperature is that quantity which, at equilibrium, is given by Eq. (10). Therefore, using that dS is an exact differential and the known transformations of the extensive quantities, E, V and N , it follows that the transformation of temperature which is in agreement with the three laws of thermodynamics is the one given by Eq. (16).

In addition, as a natural consequence of our treatment, we found that the pressure transformation is given by Eq. (19). This transformation does not agree with the one stated by many authors, which consider pressure to be Lorentz invariant. Nevertheless the transformation given by Eq. (19) must be the correct one in a relativistic thermodynamics theory because it corresponds to the definition of pressure given by thermodynamics, as is shown in Eq. (17).

Finally we showed that, if we accept our transformations for the pressure, temperature, number of particles and volume, the equation of state of an ideal gas is Lorentz invariant, which is in accord with the first principle of special relativity. It should be noted that if we have taken the pressure and temperature as Lorentz invariants the equation of state of an ideal gas would not be satisfied. Also, if we had considered the pressure as Lorentz invariant and the transformation of temperature as $T' = \gamma^{-1}T$ the equation of state of an ideal gas would have been satisfied but not the third law of thermodynamics.

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